

New massive spin two model on curved space-time

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We have proposed a new ghost-free model with interactions of massive spin two particles in Phys. Rev. D **90** (2014) 043006 [arXiv:1402.5737 [hep-th]]. Although the model is ghost-free on the Minkowski space-time, it is not obvious whether or not this desirable property is preserved on curved space-time. In fact, Buchbinder et al. already pointed out that the Fierz-Pauli theory is not ghost-free on curved space-time without non-minimal coupling terms. In this paper, we construct a new theory of massive spin two particles with non-minimal coupling on curved space-time and show that the model can be ghost-free. Furthermore, we propose new non-minimal coupling terms.

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I. INTRODUCTION

The theory of massive spin two particles has a long history. Fierz and Pauli studied the theories with arbitrary spin and succeeded in formulating the model describing the free massive spin two particle in 1939 [1]. This theory is well known as the Fierz-Pauli theory. Although the model is a free field theory, the construction is non-trivial because the mass term generally leads to a ghost and breaks the consistency as a quantum theory. Fierz and Pauli removed the ghost by tuning the relative values between the coefficients of the non-derivative quadratic terms. Since the Fierz-Pauli theory has already lost the gauge symmetry due to the mass term, it might be expected that arbitrary interactions could be allowed in the massive spin two field theory unlike in the massless theory. Boulware and Deser [2], however, suggested that non-linear terms generally generate another kind of ghost called the Boulware-Deser ghost. In addition to the ghost, another problem appears if we regard the massive spin two theory as an alternative theory of gravity. The prediction of the free massive spin two theory does not coincide with the free massless theory even in the massless limit. This fact was pointed out by van Dam, Veltman and Zakharov (vDVZ) [3] although the discontinuity can be screened by some non-linear effects called the Vainshtein mechanism [4]. (see, for example, Ref. [5]). After the work by Boulware and Deser, the studies of non-linear massive spin two theories had not progressed until 2002 because the appearance of the Boulware-Deser ghost suggested some kind of no-go theorem.

In 2002, Arkani-Hamed, Georgi, and Schwartz [6] considered massive gravity as a low energy effective field theory and showed that some class of the infinite potential terms can make the cut-off scale larger. Eight years later, de Rham, Gabadadze, and Tolley succeeded in obtaining the formal expression of the potential terms [7] and it was proved that the potential-tuned theory is ghost-free [8, 10]. This theory is called the dRGT massive gravity. The most essential part for the ghost-free property is the characteristic forms of the fully non-linear potential terms. As the dRGT massive gravity is accompanied with a non-dynamical metric called fiducial metric, we may consider to make the fiducial metric dynamical and obtain theories called bigravity which contain two dynamical metrics [9, 11, 12].

Recently Hinterbichler [14] suggested the possibility of new derivative interaction terms in the dRGT massive gravity by showing the existence of ghost-free derivative interactions for the Fierz-Pauli theory. It was shown that the leading term of the potential in the dRGT is also ghost-free as for the Fierz-Pauli theory. Based on this discussion, we constructed a new massive spin two model [15] which contains a kinetic term and potential terms only.

Although the new model is ghost-free on Minkowski space-time, it is not obvious whether or not the model on curved space-time keeps the property. The Fierz-Pauli theory coupled with gravity has been already discussed by Buchbinder et al. and they revealed that the minimal coupling model is not ghost-free [16]. This is because some additional terms including the curvature tensor appear from non-commutativity of covariant derivatives and prevent the construction of the constraint. Therefore, non-minimal coupling terms are necessary so that the free massive spin two theory does not include any ghost when coupled with gravity. As a result, they formulated the ghost-free Fierz-Pauli theory on non-trivial background by adding two terms with non-minimal coupling and restricting the background to be the Einstein manifold. This means our new model should contain at least two non-minimal couplings on curved space-time.

In this paper, we consider the new massive spin two model coupled with gravity and add the two non-minimal coupling terms to the model. Then, we study whether or not the non-minimal coupling model is ghost-free using the Lagrangian formalism. Furthermore, we investigate another possibility of new ghost-free terms on the Einstein manifold and find a new class of ghost-free potential terms while it is shown to be impossible, in our formulation, to introduce derivative interactions without any ghost.

II. NEW MODEL OF MASSIVE SPIN TWO PARTICLE

In this section, we review on the model of the massive spin two particle with interaction proposed in [15]. We start with the Lagrangian of the Fierz-Pauli theory [1]:

$$\mathcal{L}_{\text{FP}} = -\frac{1}{2}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} + \partial_\mu h_{\nu\lambda}\partial^\nu h^{\mu\lambda} - \partial_\mu h^{\mu\nu}\partial_\nu h + \frac{1}{2}\partial_\lambda h\partial^\lambda h - \frac{1}{2}m^2(h_{\mu\nu}h^{\mu\nu} - h^2). \quad (1)$$

In order to eliminate the ghost, we need to tune the relative sign of the mass term. By Hinterbichler [14], it has been pointed out that we may add new interaction terms to this model without generating any ghost by the specific linear combinations of the interaction terms. In four dimensions, we know that only two kinds of non-derivative interactions exist:

$$\mathcal{L}_3 \sim \eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3} h_{\mu_1\nu_1} h_{\mu_2\nu_2} h_{\mu_3\nu_3}, \quad (2)$$

$$\mathcal{L}_4 \sim \eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4} h_{\mu_1\nu_1} h_{\mu_2\nu_2} h_{\mu_3\nu_3} h_{\mu_4\nu_4}. \quad (3)$$

Here $\eta^{\mu_1\nu_1\cdots\mu_n\nu_n}$ is the product of n $\eta_{\mu\nu}$ given by anti-symmetrizing the indices ν_1, ν_2, \dots , and ν_n . Some examples are given by,

$$\begin{aligned} \eta^{\mu_1\nu_1\mu_2\nu_2} &\equiv \eta^{\mu_1\nu_1}\eta^{\mu_2\nu_2} - \eta^{\mu_1\nu_2}\eta^{\mu_2\nu_1}, \\ \eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3} &\equiv \eta^{\mu_1\nu_1}\eta^{\mu_2\nu_2}\eta^{\mu_3\nu_3} - \eta^{\mu_1\nu_1}\eta^{\mu_2\nu_3}\eta^{\mu_3\nu_2} + \eta^{\mu_1\nu_2}\eta^{\mu_2\nu_3}\eta^{\mu_3\nu_1} \\ &\quad - \eta^{\mu_1\nu_2}\eta^{\mu_2\nu_1}\eta^{\mu_3\nu_3} + \eta^{\mu_1\nu_3}\eta^{\mu_2\nu_1}\eta^{\mu_3\nu_2} - \eta^{\mu_1\nu_3}\eta^{\mu_2\nu_2}\eta^{\mu_3\nu_1}. \end{aligned} \quad (4)$$

The detailed property of $\eta^{\mu_1\nu_1\cdots\mu_n\nu_n}$ is summarized in Appendix A. In [15], it was proposed a new model of massive spin two particles by adding the terms in (2) and (3) to the Fierz-Pauli Lagrangian in (1),

$$\begin{aligned} \mathcal{L}_{h0} &= -\frac{1}{2}\eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3} (\partial_{\mu_1}\partial_{\nu_1}h_{\mu_2\nu_2})h_{\mu_3\nu_3} + \frac{m^2}{2}\eta^{\mu_1\nu_1\mu_2\nu_2}h_{\mu_1\nu_1}h_{\mu_2\nu_2} \\ &\quad - \frac{\mu}{3!}\eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}h_{\mu_1\nu_1}h_{\mu_2\nu_2}h_{\mu_3\nu_3} - \frac{\lambda}{4!}\eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}h_{\mu_1\nu_1}h_{\mu_2\nu_2}h_{\mu_3\nu_3}h_{\mu_4\nu_4} \\ &= -\frac{1}{2}(h\Box h - h^{\mu\nu}\Box h_{\mu\nu} - h\partial^\mu\partial^\nu h_{\mu\nu} - h_{\mu\nu}\partial^\mu\partial^\nu h + 2h_\nu{}^\rho\partial^\mu\partial^\nu h_{\mu\rho}) \\ &\quad + \frac{m^2}{2}(h^2 - h_{\mu\nu}h^{\mu\nu}) - \frac{\mu}{3!}(h^3 - 3hh_{\mu\nu}h^{\mu\nu} + 2h_\mu{}^\nu h_\nu{}^\rho h_\rho{}^\mu) \\ &\quad - \frac{\lambda}{4!}(h^4 - 6h^2h_{\mu\nu}h^{\mu\nu} + 8hh_\mu{}^\nu h_\nu{}^\rho h_\rho{}^\mu - 6h_\mu{}^\nu h_\nu{}^\rho h_\rho{}^\sigma h_\sigma{}^\mu + 3(h_{\mu\nu}h^{\mu\nu})^2). \end{aligned} \quad (5)$$

Here m and μ are parameters with the dimension of mass although the parameter λ is dimensionless. The parameter μ is always takes positive values but the sign of λ is non trivial for the stabilities.

We should note that the model with cubic interactions, including the derivatives interactions, was first proposed in [18] before [14] and it was also proved that there is no ghost in the model.

III. PSEUDO-LINEAR THEORY ON FLAT SPACE

In this paper, we consider the model where the massive spin two field couples with gravity but in order to construct the model without ghost on the curved space-time, we begin with the counting of the degrees of freedom on the flat space-time by using the lagrangian formalism as in [16].

First, just for simplicity, we only include the cubic interactions, that is, $\lambda = 0$ in (5).

$$\mathcal{L} = \frac{1}{2}\eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}\partial_{\mu_1}h_{\mu_2\nu_2}\partial_{\nu_1}h_{\mu_3\nu_3} + \frac{m^2}{2}\eta^{\mu_1\nu_1\mu_2\nu_2}h_{\mu_1\nu_1}h_{\mu_2\nu_2} - \frac{\mu}{3!}\eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}h_{\mu_1\nu_1}h_{\mu_2\nu_2}h_{\mu_3\nu_3}. \quad (6)$$

By the variations with respect to $h_{\mu\nu}$, we obtain following equations,

$$0 = E_{\mu\nu} = -\eta_{(\mu\nu)\mu_1\nu_1\mu_2\nu_2}\partial^{\mu_1}\partial^{\nu_1}h^{\mu_2\nu_2} + m^2\eta_{\mu\nu\mu_1\nu_1}h^{\mu_1\nu_1} - \frac{\mu}{2}\eta_{(\mu\nu)\mu_1\nu_1\mu_2\nu_2}h^{\mu_1\nu_1}h^{\mu_2\nu_2}. \quad (7)$$

In the equations (7), there are equations which contain the first order derivative with respect to time but do not contain the second order derivative,

$$0 = E_{0\nu} = -\eta_{(0\nu)\mu_1\nu_1\mu_2\nu_2}\partial^{\mu_1}\partial^{\nu_1}h^{\mu_2\nu_2} + m^2\eta_{0\nu\mu_1\nu_1}h^{\mu_1\nu_1} - \frac{\mu}{2}\eta_{(0\nu)\mu_1\nu_1\mu_2\nu_2}h^{\mu_1\nu_1}h^{\mu_2\nu_2}. \quad (8)$$

Because of the anti-symmetry of $\eta_{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}$, these equations do not contain any term including the second order derivatives with respect to time, $\ddot{h}_{\mu\nu}$ nor the first order derivatives of h_{00} with respect to time. Then we may regard $h_{0\mu}$ as auxiliary fields. The remaining equations in (7) contains the second order derivative with respect to time,

$$0 = E_{ij} = \eta_{(ij)kl} \ddot{h}_{kl} + (\text{terms without } \ddot{h}). \quad (9)$$

Here we used the following identity,

$$\eta_{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3} = \eta_{\mu_1\nu_1}\eta_{\mu_2\nu_2\mu_3\nu_3} + \eta_{\mu_1\nu_2}\eta_{\mu_2\nu_3\mu_3\nu_1} + \eta_{\mu_1\nu_3}\eta_{\mu_2\nu_1\mu_3\nu_2}. \quad (10)$$

For the convenience in the argument later, we now solve Eq. (9) with respect to \ddot{h}_{ij} . Because the inverse of the coefficient matrix $A_{ij,kl} \equiv \eta_{(ij)kl}$ in (9) is given by

$$A^{-1}_{kl,mn} = -\eta_{m(k}\eta_{l)n} + \frac{1}{2}\eta_{kl}\eta_{mn}, \quad (11)$$

\ddot{h}_{ij} can be expressed by using the terms which do not contain the second order derivative with respect to time, as follows,

$$0 = \left(-\eta_{m(i}\eta_{j)n} + \frac{1}{2}\eta_{ij}\eta_{mn} \right) E_{ij} = \ddot{h}_{mn} + (\text{terms without } \ddot{h}). \quad (12)$$

In order to count the degrees of freedom of this system (that is, the number of independent initial values in classical mechanics), we consider the conditions that the equations in (8), which only contains the terms including the first order derivative with respect to time, is conserved, that is, the equations are invariant under the translation of time. Then we may regard $h_{0\mu}$ as dynamical fields in the equations obtained from the condition. Because the obtained equations guarantee the conservation of the original equations (8), we may forget the equations except for the initial conditions. Thus the original equations can be regarded as the ‘‘constraints’’ on the initial values. We continue this procedure until the conditions become the second order differential equations of $h_{0\mu}$ with respect to time.

Now, we regard the equations (8) as primary constraints:

$$0 \approx E_{0\nu} \equiv \phi_{\nu}^{(1)}. \quad (13)$$

Here \approx means equivalence up to constraints, and the constraint (13) does not guarantee that the equation is invariant under the time evolution. From the conservation of $E_{0\mu} = \phi_{\mu}^{(1)}$, by using the equations (9) and the constraints (8), we obtain

$$0 = -\dot{\phi}_{\nu}^{(1)} \approx m^2 \eta_{(\mu\nu)\mu_1\nu_1} \partial^{\mu} h^{\mu_1\nu_1} - \mu \eta_{(\mu\nu)\mu_1\nu_1\mu_2\nu_2} \partial^{\mu} h^{\mu_1\nu_1} \cdot h^{\mu_2\nu_2} = \partial^{\mu} E_{\mu\nu} \equiv \phi_{\nu}^{(2)}. \quad (14)$$

The derivation of the above equation is a little bit tedious but if we use the equation,

$$\partial^{\mu} E_{\mu\nu} = m^2 \eta_{(\mu\nu)\mu_1\nu_1} \partial^{\mu} h^{\mu_1\nu_1} - \mu \eta_{(\mu\nu)\mu_1\nu_1\mu_2\nu_2} \partial^{\mu} h^{\mu_1\nu_1} \cdot h^{\mu_2\nu_2}, \quad (15)$$

from the beginning (indeed the above equation (15) can be obtained trivially by using the anti-symmetry of η tensor), we find

$$-E_{i\nu,i} + m^2 \eta_{(\mu\nu)\mu_1\nu_1} \partial^{\mu} h^{\mu_1\nu_1} - \mu \eta_{(\mu\nu)\mu_1\nu_1\mu_2\nu_2} \partial^{\mu} h^{\mu_1\nu_1} \cdot h^{\mu_2\nu_2} = -\dot{E}_{0\nu} = -\dot{\phi}_{\nu}^{(1)}. \quad (16)$$

The terms $E_{i\nu,i}$ can be ignored up to the equations (9) and the constraints (8). Thus by using the conservation of $\phi_{\nu}^{(1)}$, we obtain the conditions $\phi_{\nu}^{(2)} = 0$, which is identical with (14) without tedious calculations. Now, the primary constraints (8) can be regarded as the condition only on the initial values. Instead of primary constraints, we can impose the condition $\phi_{\nu}^{(2)} = 0$ for any time. The primary constraints hold automatically thanks to the conditions for the conservation of the constraints in time $\phi_{\nu}^{(2)} = 0$. Because, however, $\phi_{\nu}^{(2)} = 0$ is the equation including only the first order differential equation with respect to the time, we have to change the equations in (14), which is defined by the strong equality $=$ to the equations defined by the weak equality \approx and we regard $\phi_{\nu}^{(2)}$ as secondary constraints $\phi_{\nu}^{(2)} \approx 0$. In order to derive the conditions for the conservation of the constraints, we use one more relation:

$$0 = \partial^{\mu} \partial^{\nu} E_{\mu\nu} + \frac{m^2}{2} \eta^{\mu\nu} E_{\mu\nu} - \mu h^{\mu\nu} E_{\mu\nu}$$

$$= -\frac{3\mu m^2}{2}\eta_{\mu\nu\mu_1\nu_1}h^{\mu\nu}h^{\mu_1\nu_1} + \frac{\mu^2}{2}\eta_{\mu\nu\mu_1\nu_1\mu_2\nu_2}h^{\mu\nu}h^{\mu_1\nu_1}h^{\mu_2\nu_2} + \frac{3m^4}{2}h - \mu\eta_{\mu\nu\mu_1\nu_1\mu_2\nu_2}\partial^\mu h^{\mu_1\nu_1}\partial^\nu h_{\mu_2\nu_2}. \quad (17)$$

Then, by using the conservation of $\phi_0^{(2)}$, we obtain

$$\begin{aligned} 0 &\approx -\dot{\phi}_0^{(2)} = -\partial_0\partial^\mu E_{\mu 0} \\ &\approx \partial^\mu\partial^\nu E_{\mu\nu} + \frac{m^2}{2}\eta^{\mu\nu}E_{\mu\nu} - \mu h^{\mu\nu}E_{\mu\nu} \\ &= -\frac{3\mu m^2}{2}\eta_{\mu\nu\mu_1\nu_1}h^{\mu\nu}h^{\mu_1\nu_1} + \frac{\mu^2}{2}\eta_{\mu\nu\mu_1\nu_1\mu_2\nu_2}h^{\mu\nu}h^{\mu_1\nu_1}h^{\mu_2\nu_2} + \frac{3m^4}{2}h - \mu\eta_{\mu\nu\mu_1\nu_1\mu_2\nu_2}\partial^\mu h^{\mu_1\nu_1}\partial^\nu h_{\mu_2\nu_2} \equiv \phi^{(3)}. \end{aligned} \quad (18)$$

Note that Eq. (18) does not contain any derivative of h_{00} and the second order derivatives of h_{0i} and h_{ij} with respect to time. On the other hand, from the conservation of the constraints $\phi_i^{(2)}$, we obtain the second order derivative equations for h_{0i} up to the equation (9),

$$\dot{\phi}_i^{(2)} = (m^2\eta_{ij} - \mu\eta_{ijkl}h_{kl})\ddot{h}_{0j} + (\text{terms without } \ddot{h}) = 0. \quad (19)$$

Therefore, we can use the equations in (19) as the equations that describe dynamics of h_{0i} . Except the special configurations of fields where the matrix $M_{ij} = m^2\eta_{ij} - \mu\eta_{ijkl}h_{kl}$ has any vanishing eigenvalue, we can solve the equations (19) with respect to \ddot{h}_{0i} as follows,

$$0 = \frac{1}{m^2} \left[\eta_{ij} + \sum_{n=1}^{\infty} (\mathbf{H}^n)_{ij} \right] \dot{\phi}_j^{(2)} = \ddot{h}_{0i} + (\text{terms without } \ddot{h}). \quad (20)$$

Here, $(\mathbf{H}^n)_{ij}$ is defined by

$$(\mathbf{H}^n)_{ij} \equiv H_{ik_1}H_{k_1k_2}\cdots H_{k_{n-1}j}, \quad H_{ij} \equiv \frac{\mu}{m^2}\eta_{ijkl}h_{kl}. \quad (21)$$

Now, let us consider the condition for the conservation of the constraint $\phi^{(3)}$. Because $\phi^{(3)}$ does not contain any derivative of h_{00} nor the second order derivative of h_{0i} , h_{ij} with respect to time, $\dot{\phi}^{(3)}$ contain the first order derivative of h_{00} and the second order derivative of h_{0i} and h_{ij} with respect to time. As shown, \ddot{h}_{ij} and \ddot{h}_{0i} can be eliminated by using Eqs. (12) and (20). Therefore we obtain one more constraint which does not contain the terms including the second order derivative with respect to time,

$$\dot{\phi}^{(3)} = (\text{terms without } \ddot{h}) \equiv \phi^{(4)} \approx 0. \quad (22)$$

Although we do not give explicit form of this constraint, we can see that the condition for the conservation of the constraint (22) does not give any more constraints, which can be found as follows. By focusing only on the linear terms, we found the condition for the conservation of the constraint $\phi^{(4)}$ is given by

$$0 = \dot{\phi}^{(4)} = \frac{3m^4}{2}\ddot{h} + \mathcal{O}(h^2). \quad (23)$$

We should note that the first term cannot be canceled by $\mathcal{O}(h^2)$ terms. In addition, because the first term contains \ddot{h}_{00} , this equation defines the dynamics of h_{00} . Therefore the condition for the conservation of $\phi^{(4)}$ does not give any more constraints. Finally, we obtain 10 equations including second order derivative with respect to time which describe the dynamics of $h_{\mu\nu}$ and 10 constraints which restrict the initial values. Then we find the pseudo-linear theory has $(20 - 10)/2 = 5$ degrees of freedom on the flat space.

IV. PSEUDO-LINEAR THEORY ON CURVED SPACE

Before the discussion of the new model on curved space-time, let us briefly review on the Fierz-Pauli theory on curved space-time. In [16], Buchbinder et al. showed that the Fierz-Pauli theory on the non-trivial background must have the non-minimal coupling terms and the space-time is required to be the Einstein manifold in order to keep the consistency. This ghost-free action is given by

$$S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2} \nabla_\mu h \nabla^\mu h - \frac{1}{2} \nabla_\mu h_{\nu\rho} \nabla^\mu h^{\nu\rho} - \nabla^\mu h_{\mu\nu} \nabla^\nu h + \nabla_\mu h_{\nu\rho} \nabla^\rho h^{\nu\mu} + \frac{m^2}{2} g^{\mu_1\nu_1\mu_2\nu_2} h_{\mu_1\nu_1} h_{\mu_2\nu_2} \right. \\ \left. + \frac{\xi}{4} R h_{\alpha\beta} h^{\alpha\beta} + \frac{1-2\xi}{8} R h^2 \right\}, \quad (24)$$

This suggests that these non-minimal coupling terms should be added to the new model (5) on curved space-time. Thus, we consider the following action.

$$S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2} \nabla_\mu h \nabla^\mu h - \frac{1}{2} \nabla_\mu h_{\nu\rho} \nabla^\mu h^{\nu\rho} - \nabla^\mu h_{\mu\nu} \nabla^\nu h + \nabla_\mu h_{\nu\rho} \nabla^\rho h^{\nu\mu} + \frac{m^2}{2} g^{\mu_1\nu_1\mu_2\nu_2} h_{\mu_1\nu_1} h_{\mu_2\nu_2} \right. \\ \left. - \frac{\mu}{3!} g^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3} h_{\mu_1\nu_1} h_{\mu_2\nu_2} h_{\mu_3\nu_3} - \frac{\lambda}{4!} g^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4} h_{\mu_1\nu_1} h_{\mu_2\nu_2} h_{\mu_3\nu_3} h_{\mu_4\nu_4} + \frac{\xi}{4} R h_{\alpha\beta} h^{\alpha\beta} + \frac{1-2\xi}{8} R h^2 \right\}, \quad (25)$$

Here the metric g is chosen to be the Einstein manifold, where the curvatures satisfy the following condition:

$$R_{\mu\nu} = \frac{R}{4} g_{\mu\nu}. \quad (26)$$

As a first step, let us consider the $\lambda = 0$ case and prove that there appear 5 degrees of freedom. By using the action (25) with the $\lambda = 0$, we find that $h_{\mu\nu}$ obeys the following equations

$$0 = E_{\mu\nu} = g^{\alpha\beta} \nabla_\alpha \nabla_\beta h_{\mu\nu} - g_{\mu\nu} g^{\alpha\beta} g^{\gamma\delta} \nabla_\alpha \nabla_\beta h_{\gamma\delta} + g_{\mu\nu} g^{\alpha\gamma} g^{\beta\delta} \nabla_\alpha \nabla_\beta h_{\gamma\delta} - 2g^{\sigma\rho} \nabla_\sigma \nabla_{(\mu} h_{\nu)\rho} + g^{\alpha\beta} \nabla_\mu \nabla_\nu h_{\alpha\beta} \\ + m^2 g_{(\mu\nu)}^{\alpha\beta} h_{\alpha\beta} + \frac{\xi}{2} R h_{\mu\nu} + \frac{1-2\xi}{4} R g^{\alpha\beta} g_{\mu\nu} h_{\alpha\beta} - \frac{\mu}{2} g_{(\mu\nu)}^{\mu_1\nu_1\mu_2\nu_2} h_{\mu_1\nu_1} h_{\mu_2\nu_2} \\ = -g_{(\mu\nu)}^{\mu_1\nu_1\mu_2\nu_2} \nabla_{\mu_1} \nabla_{\nu_1} h_{\mu_2\nu_2} + (\text{terms without } \nabla\nabla h) \\ = -g_{i(\mu} g_{\nu)j} g^{ij00\mu_2\nu_2} \nabla_0 \nabla_0 h_{\mu_2\nu_2} + (\text{terms without } \nabla_0 \nabla_0 h). \quad (27)$$

In this section, we regard the equations which do not include $\nabla_0 \nabla_0 h$ (or $\partial_0 \partial_0 h$) as constraints. Because $E_{\mu\nu}$ can be expressed as in the last line of (27), we find that $E_{0\mu}$ contain the second order derivative terms with respect to time. We should note, however, that these terms including the second order derivatives with respect to time can be eliminated by the linear combinations of $E_{\mu\nu}$ as follows,

$$E^0{}_\nu = g^{00} E_{0\nu} + g^{0i} E_{i\nu} \\ = -g_{\nu\sigma} g^{(0\sigma)\mu_1\nu_1\mu_2\nu_2} \nabla_{\mu_1} \nabla_{\nu_1} h_{\mu_2\nu_2} + (\text{terms without } \nabla\nabla h) \\ = (\text{terms without } \nabla_0 \nabla_0 h) \equiv \phi_\nu^{(1)} \approx 0. \quad (28)$$

Therefore, $\phi_\nu^{(1)} \equiv E^0{}_\nu$ can be regarded as primary constraints. Then (ij) -components of Eq. (27) have the following forms:

$$0 = E_{ij} = -g_{m(i} g_{j)n} g^{mn00kl} \nabla_0 \nabla_0 h_{kl} + (\text{terms without } \nabla_0 \nabla_0 h). \quad (29)$$

In order to solve Eq. (29) with respect to $\nabla_0 \nabla_0 h_{ij}$, we use the ADM decomposition:

$$g^{00} = -\frac{1}{N^2}, \quad g_{0k} = N_k, \quad g_{ij} = e_{ij}, \\ g_{00} = N^k N_k - N^2, \quad g^{0i} = \frac{N^i}{N^2}, \quad g^{ij} = e^{ij} - \frac{N^i N^j}{N^2}. \quad (30)$$

Here e_{ij} is a three dimensional metric field and has the following properties,

$$e^{ij} e_{ij} = \delta^i{}_j, \quad N^i \equiv e^{ij} N_j, \quad e^{ij} = g^{ij} - \frac{g^{0i} g^{0j}}{g^{00}}. \quad (31)$$

By using the ADM decomposition, the coefficient matrix in equations (29) can be expressed as (see (A8) in Appendix),

$$A_{ij}{}^{,kl} \equiv -g_{m(i} g_{j)n} g^{mn00kl} = \frac{1}{N^2} e_{(ij)}{}^{kl}. \quad (32)$$

Here

$$e_{i_1 j_1 i_2 j_2 \dots i_n j_n} \equiv e_{i_1 j_1} e_{i_2 j_2} \dots e_{i_n j_n} - e_{i_1 j_2} e_{i_2 j_1} \dots e_{i_n j_n} + \dots \quad (33)$$

We now define the raising and lowering the indices in $e_{i_1 j_1 \dots i_n j_n}$ by using e^{ij} and e_{ij} . The inverse matrix of the coefficient matrix (32) is expressed as

$$A^{-1}_{kl},{}^{mn} = N^2 \left(\frac{1}{2} e_{kl} e^{mn} - e_{(k}{}^m e_{l)}{}^n \right) = \frac{1}{g^{00}} \left\{ g_{(k}{}^m g_{l)}{}^n - \frac{1}{2} g_{kl} (g^{mn} - \frac{g^{0m} g^{0n}}{g^{00}}) \right\},$$

$$A_{ij},{}^{kl} A^{-1}_{kl},{}^{mn} = \delta_{(i}^k \delta_{j)}^l. \quad (34)$$

Then, we can solve Eq. (29) with respect to $\nabla_0 \nabla_0 h_{ij}$ as follows,

$$0 = \frac{1}{g^{00}} \left\{ g_{(k}{}^i g_{l)}{}^j - \frac{1}{2} g_{kl} \left(g^{ij} - \frac{g^{0i} g^{0j}}{g^{00}} \right) \right\} E_{ij} = \nabla_0 \nabla_0 h_{ij} + (\text{terms without } \nabla_0 \nabla_0 h). \quad (35)$$

Because Eq. (29) gives 6 independent equations including the second order derivative with respect to time and these equations are also independent of the primary constraints $\phi_\nu^{(1)}$, these equations describe the dynamics of h_{ij} . In order to obtain the conditions for the conservations of the primary constraints, we use the following relations:

$$\begin{aligned} \nabla^\mu E_{\mu\nu} &= \frac{R}{4} g^{\alpha\beta} \nabla_\nu h_{\alpha\beta} - \frac{R}{2} g^{\sigma\rho} \nabla_\sigma h_{\rho\nu} + m^2 g_{\nu\nu_1} g^{\mu_1\nu_1\mu_2\nu_2} \nabla_{\mu_1} h_{\mu_2\nu_2} \\ &\quad + \frac{\xi}{2} R g^{\sigma\rho} \nabla_\sigma h_{\rho\nu} + \frac{1-2\xi}{4} R g^{\alpha\beta} \nabla_\nu h_{\alpha\beta} - \mu g_{\nu\nu_1} g^{(\mu_1\nu_1)\mu_2\nu_2\mu_3\nu_3} (\nabla_{\mu_1} h_{\mu_2\nu_1}) h_{\mu_2\nu_2} \\ &= \left(\frac{1-\xi}{2} R + m^2 \right) g_{\nu\nu_1} g^{\mu_1\nu_1\mu_2\nu_2} \nabla_{\mu_1} h_{\mu_2\nu_2} - \mu g_{\nu\nu_1} g^{(\mu_1\nu_1)\mu_2\nu_2\mu_3\nu_3} (\nabla_{\mu_1} h_{\mu_2\nu_2}) h_{\mu_3\nu_3}. \end{aligned} \quad (36)$$

Then the secondary constraints are obtained as

$$\partial_0 \phi_\nu^{(1)} = \partial_0 E_\nu^0 \approx \nabla^\mu E_{\mu\nu} \equiv \phi_\nu^{(2)} \approx 0. \quad (37)$$

For convenience, we choose independent constraints as follows,

$$\phi^{(2)0} \equiv g^{00} \phi_0^{(2)} + g^{0i} \phi_i^{(2)} \approx 0, \quad \phi_i^{(2)} \approx 0. \quad (38)$$

Furthermore, by using the following relation:

$$\begin{aligned} \nabla^\mu \nabla^\nu E_{\mu\nu} &+ \frac{m^2}{2} g^{\mu\nu} E_{\mu\nu} - \mu h^{\mu\nu} E_{\mu\nu} + \frac{1-\xi}{4} R g^{\mu\nu} E_{\mu\nu} \\ &= h \left(\frac{3m^4}{2} + \frac{5-6\xi}{4} m^2 R + \frac{(1-\xi)(2-3\xi)}{8} R^2 \right) \\ &\quad - \frac{3\mu m^2}{2} g^{\mu_1\nu_1\mu_2\nu_2} h_{\mu_1\nu_1} h_{\mu_2\nu_2} - \mu g^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3} (\nabla_{\mu_1} h_{\mu_2\nu_2}) \nabla_{\nu_1} h_{\mu_3\nu_3} \\ &\quad + \frac{\mu^2}{2} g^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3} h_{\mu_1\nu_1} h_{\mu_2\nu_2} h_{\mu_3\nu_3} - \frac{7-9\xi}{12} \mu R g^{\mu_1\nu_1\mu_2\nu_2} h_{\mu_1\nu_1} h_{\mu_2\nu_2} - \mu C^{\mu\alpha\nu\beta} h_{\mu\nu} h_{\alpha\beta}, \end{aligned} \quad (39)$$

we find one more constraint:

$$\partial_0 \phi^{(2)0} \approx \nabla^\mu \nabla^\nu E_{\mu\nu} + \frac{m^2}{2} g^{\mu\nu} E_{\mu\nu} - \mu h^{\mu\nu} E_{\mu\nu} + \frac{1-\xi}{4} R g^{\mu\nu} E_{\mu\nu} \equiv \phi^{(3)} \approx 0. \quad (40)$$

We have to note that the non-minimal coupling terms in (36) play a very important role for the existence of the constraint $\phi^{(3)}$. The term $\frac{R}{4} g^{\alpha\beta} \nabla_\nu h_{\alpha\beta} - \frac{R}{2} g^{\sigma\rho} \nabla_\sigma h_{\rho\nu}$ in (36) contains the derivatives of h_{00} with respect to time and this prevents us from having the appropriate number of constraints. The contribution from the non-minimal couplings, however, cancels out these time-derivative terms and enables the system to have 5 degrees of freedom.

On the other hand, the term including the curvature tensor does not appear from the cubic potential in $\nabla^\mu E_{\mu\nu}$ and, as a result, does not contain any derivative of h_{00} with respect to time. Needless to say, the derivative of h_{00} with respect to time emerges when we act another covariant derivative on $\nabla^\mu E_{\mu\nu}$, but this term is also eliminated by the term $h^{\mu\nu} E_{\mu\nu}$ in (40). This means that we do not need any additional non-minimal coupling terms for this system

to be ghost-free. Generally, we can add a new non-minimal coupling term $Rg^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}h_{\mu_1\nu_1}h_{\mu_2\nu_2}h_{\mu_3\nu_3}$ without any ghost, but this fact does not change the following analysis. This is because the scalar curvature is constant on the Einstein manifold and the effect of the new term can be absorbed by the redefinition of μ . On the other hand, by using the ADM decomposition, the conditions for the conservation of $\phi_i^{(2)}$ have the following forms:

$$\begin{aligned}\partial_0\phi_i^{(2)} &\approx \nabla_0\nabla^\mu E_{\mu i} = B_i{}^j\nabla_0\nabla_0 h_{0j} + C_i{}^{kl}\nabla_0\nabla_0 h_{kl} + (\text{terms without } \nabla_0\nabla_0 h) = 0, \\ B_i{}^j &\equiv \frac{1}{N^2} \left[\left(\frac{1-\xi}{2} R + m^2 \right) \delta_i{}^j - \mu e_i{}^{mn} h_{mn} \right], \\ C_i{}^{kl} &\equiv -\frac{1}{N^2} \left[\left(\frac{1-\xi}{2} R + m^2 \right) N^k \delta_i{}^l + \mu \left\{ e_i{}^{kl} h_{0j} + N^k e_i{}^{lmn} h_{mn} + N^m e_i{}^{nkl} h_{mn} \right\} \right].\end{aligned}\quad (41)$$

Here we have used (A7) and (A8). Then $\nabla_0\nabla_0 h_{kl}$ can be eliminated by using Eq. (35) as follows,

$$\partial_0\phi_i^{(2)} \approx \nabla_0\nabla^\mu E_{\mu i} - C_i{}^{kl} A^{-1}{}_{kl}{}^{mn} E_{mn} = B_i{}^a \nabla_0\nabla_0 h_{0a} + (\text{terms without } \nabla_0\nabla_0 h) = 0. \quad (42)$$

Thus, we find that the conditions (41) describe the dynamics of h_{0i} so that the constraints in (39) are conserved. Except the special configuration of field where the matrix $B_i{}^j$ has vanishing eigenvalue, we can solve the equations in (41) with respect to $\nabla_0\nabla_0 h_{0i}$ as follows,

$$\begin{aligned}B^{-1}{}_k{}^i \left[\nabla_0\nabla^\mu E_{\mu i} - C_i{}^{kl} A^{-1}{}_{kl}{}^{mn} E_{mn} \right] &= \nabla_0\nabla_0 h_{0k} + (\text{terms without } \nabla_0\nabla_0 h) = 0, \\ B^{-1}{}_i{}^j &= \frac{N^2}{\frac{1-\xi}{2} R + m^2} \left[\delta_i{}^j + \sum_{n=1}^{\infty} (\mathbf{H}^n)_i{}^j \right], \\ (\mathbf{H}^n)_i{}^j &\equiv H_{ik_1} e^{k_1 l_1} H_{l_1 k_2} e^{k_2 l_2} \dots H_{l_{n-1} k_n} e^{k_n j}, \quad H_{ij} \equiv \frac{\mu}{\frac{1-\xi}{2} R + m^2} e_{ij}{}^{mn} h_{mn}.\end{aligned}\quad (43)$$

As in the case of the flat background, the constraint obtained from the condition for the conservation of $\phi^{(3)}$ has the following form:

$$\partial_0\phi^{(3)} \approx (\text{terms without } \nabla_0\nabla_0 h) \equiv \phi^{(4)} \approx 0. \quad (44)$$

By focusing the linear terms, we find that the condition for the conservation of the constraint $\phi^{(4)}$ defines the dynamics of h_{00} . As a result, the pseudo-linear theory described by action (25) has 5 degrees of freedom on the Einstein manifold (26).

V. $\lambda \neq 0$ CASE

We now investigate if the arguments in the previous sections can be extended to the $\lambda \neq 0$ case. In the $\lambda \neq 0$ case, the equations are modified as follows,

$$\begin{aligned}0 = E_{\mu\nu} &= g^{\alpha\beta} \nabla_\alpha \nabla_\beta h_{\mu\nu} - g_{\mu\nu} g^{\alpha\beta} g^{\gamma\delta} \nabla_\alpha \nabla_\beta h_{\gamma\delta} + g_{\mu\nu} g^{\alpha\gamma} g^{\beta\delta} \nabla_\alpha \nabla_\beta h_{\gamma\delta} - 2g^{\sigma\rho} \nabla_\sigma \nabla_\rho h_{\mu\nu} \\ &\quad + g^{\alpha\beta} \nabla_\mu \nabla_\nu h_{\alpha\beta} + m^2 g_{(\mu\nu)}{}^{\alpha\beta} h_{\alpha\beta} + \frac{\xi}{2} R h_{\mu\nu} + \frac{1-2\xi}{4} R g^{\alpha\beta} g_{\mu\nu} h_{\alpha\beta} \\ &\quad - \frac{\mu}{2} g_{(\mu\nu)}{}^{\mu_1\nu_1\mu_2\nu_2} h_{\mu_1\nu_1} h_{\mu_2\nu_2} - \frac{\lambda}{3!} g_{(\mu\nu)}{}^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3} h_{\mu_1\nu_1} h_{\mu_2\nu_2} h_{\mu_3\nu_3}.\end{aligned}\quad (45)$$

Then we obtain the following primary constraint:

$$E^0{}_\nu \equiv \phi_\nu^{(1)} \approx 0. \quad (46)$$

By using the conservation of the constraint $\phi_\nu^{(1)}$ (46), we find that the secondary constraints are given by

$$\begin{aligned}\nabla^\mu E_{\mu\nu} &= \left(\frac{1-\xi}{2} R + m^2 \right) g_{\nu\nu_1} g^{\mu_1\nu_1\mu_2\nu_2} \nabla_{\mu_1} h_{\mu_2\nu_2} - \mu g_{\nu\nu_1} g^{(\mu_1\nu_1)\mu_2\nu_2\mu_3\nu_3} (\nabla_{\mu_1} h_{\mu_2\nu_2}) h_{\mu_3\nu_3} \\ &\quad - \frac{\lambda}{2} g_{\nu\nu_1} g^{(\mu_1\nu_1)\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4} (\nabla_{\mu_1} h_{\mu_2\nu_2}) h_{\mu_3\nu_3} h_{\mu_4\nu_4} \equiv \phi_\nu^{(2)} \approx 0.\end{aligned}\quad (47)$$

As in the $\lambda = 0$ case, the conservation of the constraints $\phi_i^{(2)}$ gives 3 equations describing the dynamics of h_{0i} . The equations of the conservation have the following structure,

$$B_i^j \nabla_0 \nabla_0 h_{j0} + (\text{terms without } \nabla_0 \nabla_0 h) = 0, \\ B_i^j \equiv \frac{1}{N^2} \left[\left(\frac{1-\xi}{2} R + m^2 \right) \delta_i^j - \mu e_i^{jmn} h_{mn} - \frac{\lambda}{2} e_{ii_1} e^{(i_1 j) i_2 j_2 i_3 j_3} h_{i_2 j_2} h_{i_3 j_3} \right]. \quad (48)$$

The matrix B_i^j , which is the coefficient of $\nabla_0 \nabla_0 h_{j0}$ can be eliminated by using the following inverse matrix:

$$B^{-1}{}_i{}^j = \frac{N^2}{\frac{1-\xi}{2} R + m^2} \left[\delta_i^j + \sum_{n=1}^{\infty} (\mathbf{H}^n)_i{}^j \right], \\ (\mathbf{H}^n)_i{}^j \equiv H_{ik_1} e^{k_1 l_1} H_{l_1 k_2} e^{k_2 l_2} \dots H_{l_{n-1} k_n} e^{k_n j}, \quad H_{ij} \equiv \frac{1}{\frac{1-\xi}{2} R + m^2} \left[\mu e_{ij}{}^{mn} h_{mn} + \frac{\lambda}{2} e_{(ij)}{}^{klmn} h_{kl} h_{mn} \right]. \quad (49)$$

Furthermore the conservation of $\phi^{(2)0}$ gives an additional constraint:

$$\nabla^\mu \nabla^\nu E_{\mu\nu} + \frac{m^2}{2} g^{\mu\nu} E_{\mu\nu} + \frac{1-\xi}{4} R g^{\mu\nu} E_{\mu\nu} - \mu h^{\mu\nu} E_{\mu\nu} + \frac{\lambda}{2} g^{00ijklmn} A_{ij}^{-1,ab} E_{ab} h_{kl} h_{mn} \\ = -\mu g^{(\mu_1 \nu_1) \mu_2 \nu_2 \mu_3 \nu_3} (\nabla_{\mu_1} h_{\mu_2 \nu_2}) \nabla_{\nu_1} h_{\mu_3 \nu_3} - \lambda g^{(\mu_1 \nu_1) \mu_2 \nu_2 \mu_3 \nu_3 \mu_4 \nu_4} (\nabla_{\mu_1} h_{\mu_2 \nu_2}) (\nabla_{\nu_1} h_{\mu_3 \nu_3}) h_{\mu_4 \nu_4} \\ - \lambda g^{(0i) \mu_2 \nu_2 \mu_3 \nu_3 \mu_4 \nu_4} (\nabla_0 \nabla_i h_{\mu_2 \nu_2}) h_{\mu_3 \nu_3} h_{\mu_4 \nu_4} + \frac{\lambda}{2} g^{00ijklmn} A_{ij}^{-1,ab} \left(-2g_{ab}{}^{(0c) \mu\nu} \nabla_0 \nabla_c h_{\mu\nu} \right) h_{kl} h_{mn} \\ + (\text{terms without any time derivatives of } h) \equiv \phi^{(3)} \approx 0. \quad (50)$$

Here $A_{ij}^{-1,kl}$ is defined by (34). By using the expression (50), we find that the derivative of $\phi^{(3)}$ with respect to time does not contain the second order derivatives of h_{00} with respect to time but the second order derivatives of h_{0i} and h_{ij} with respect to time. Since the second order derivatives of h_{0i} and h_{ij} with respect to time can be eliminated as in the $\lambda = 0$ case, there appears one more constraint:

$$\phi^{(4)} \approx 0. \quad (51)$$

Therefore, even in the case of $\lambda \neq 0$, the pseudo-linear theory has 5 degrees of freedom on the Einstein manifold.

VI. A NEW NON-MINIMAL COUPLING TERM

In [16], in order to eliminate a ghost, non-minimal coupling terms were added to the Fierz-Pauli action. In this section, we show that there is another kind of non-minimal coupling which does not induce the ghost. We should note that the constraint $\phi^{(3)}$ is essential to exclude the extra degrees of freedom.

We now assume the quadratic part in the action to be more general form than that in [16] on the Einstein manifold, as follows,

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2} g^{\mu_1 \nu_1 \mu_2 \nu_2 \mu_3 \nu_3} \nabla_{\mu_1} h_{\mu_2 \nu_2} \nabla_{\nu_1} h_{\mu_3 \nu_3} + \frac{m^2}{2} g^{\mu_1 \nu_1 \mu_2 \nu_2} h_{\mu_1 \nu_1} h_{\mu_2 \nu_2} \right. \\ \left. + \frac{\alpha}{2} R h^2 + \frac{\beta}{2} R h_{\mu\nu} h^{\mu\nu} + \frac{\gamma}{2} C^{\mu\alpha\nu\beta} h_{\mu\nu} h_{\alpha\beta} \right], \quad (52)$$

and find the combinations of the parameters which do not induce the ghost. We should note that the kinetic term in (52) is not identical with that in (24) because the non-commutativity of the covariant derivatives induces the curvature tensor. That is, the first term in (52) is expanded as follows:

$$\frac{1}{2} g^{\mu_1 \nu_1 \mu_2 \nu_2 \mu_3 \nu_3} \nabla_{\mu_1} h_{\mu_2 \nu_2} \nabla_{\nu_1} h_{\mu_3 \nu_3} = \frac{1}{2} \nabla_\mu h \nabla^\mu h - \frac{1}{2} \nabla_\mu h_{\nu\rho} \nabla^\mu h^{\nu\rho} - \nabla^\mu h_{\mu\nu} \nabla^\nu h + \nabla_\mu h_{\nu\rho} \nabla^\rho h^{\nu\mu} \\ + \frac{R}{4} h_{\alpha\beta} h^{\alpha\beta} - \frac{R}{8} h^2 - \frac{1}{2} C^{\mu\alpha\nu\beta} h_{\mu\nu} h_{\alpha\beta} + \frac{R}{12} g^{\mu_1 \nu_1 \mu_2 \nu_2} h_{\mu_1 \nu_1} h_{\mu_2 \nu_2} \quad (53)$$

Here we have used the following relation between the Riemann curvature $R_{\mu\alpha\nu\beta}$ and the Weyl tensor $C_{\mu\alpha\nu\beta}$ on the Einstein manifold (26),

$$R_{\mu\alpha\nu\beta} = C_{\mu\alpha\nu\beta} + \frac{R}{12} g_{\mu\nu\alpha\beta}. \quad (54)$$

Thus, we have to subtract the contribution from the terms including the curvature tensor when we use $g^{\mu_1\nu_1\mu_2\nu_2\cdots}$ to express the kinetic term. In (52), these extra terms are included in the remaining non-minimal coupling terms because, on the Einstein manifold, the Riemann tensor can be decomposed into the Weyl tensor and the Ricci scalar and the Ricci tensor can be expressed in terms of the Ricci scalar.

The contribution to the equation from the kinetic terms in the action (52) is given by

$$\begin{aligned} E_K^{\mu\nu} &\equiv -g^{(\mu\nu)\mu_1\nu_1\mu_2\nu_2}\nabla_{\mu_1}\nabla_{\nu_1}h_{\mu_2\nu_2} \\ &= -g^{\mu\nu\mu_1\nu_1\mu_2\nu_2}\nabla_{\mu_1}\nabla_{\nu_1}h_{\mu_2\nu_2} + \frac{1}{2}g^{[\mu\nu]\mu_1\nu_1\mu_2\nu_2}R_{\nu_1}{}^{\sigma}{}_{\mu_1\mu_2}h_{\sigma\nu_2}. \end{aligned} \quad (55)$$

Here we have used

$$R_{\lambda\alpha\beta\gamma} + R_{\lambda\beta\gamma\alpha} + R_{\lambda\gamma\alpha\beta} = 0. \quad (56)$$

By using (54) and the following identities,

$$\begin{aligned} g^{\mu\nu\mu_1\nu_1\mu_2\nu_2}C_{\nu_1}{}^{\sigma}{}_{\mu_1\mu_2} &= (g^{\mu_1\nu_1}g^{\mu\nu\mu_2\nu_2} + g^{\mu\nu_1}g^{\mu_2\nu\mu_1\nu_2} + g^{\mu_2\nu_1}g^{\mu_1\nu\mu_2\nu_2})C_{\nu_1}{}^{\sigma}{}_{\mu_1\mu_2} \\ &= 2C^{\mu\sigma\nu_2\nu}, \\ g^{\mu\nu\mu_1\nu_1\mu_2\nu_2}g_{\nu_1\mu_1}{}^{\sigma}{}_{\mu_2} &= 2g^{\mu\nu\mu_1\nu_1\mu_2\nu_2}g_{\nu_1\mu_1}g_{\mu_2}^{\sigma} = 4g^{\mu\nu\sigma\nu_2}, \end{aligned} \quad (57)$$

we find that there is a symmetry with respect of the exchange of the indices μ and ν and therefore the last term in (55) vanishes. Then we obtain the following expression,

$$\begin{aligned} \nabla_{\mu}E_K^{\mu\nu} &= \frac{1}{2}g^{\mu\nu\mu_1\nu_1\mu_2\nu_2}R_{\nu_1}{}^{\sigma}{}_{\mu_1\mu_2}[\nabla_{\sigma}h_{\mu_2\nu_2} - \nabla_{\nu_2}h_{\mu_2\sigma}] \\ &= -C^{\mu\alpha\nu\beta}\nabla_{\mu}h_{\alpha\beta} + \frac{R}{6}g^{\mu\nu\alpha\beta}\nabla_{\mu}h_{\alpha\beta}. \end{aligned} \quad (58)$$

Here in the first line, we have used $g^{\mu\nu\mu_1\nu_1\mu_2\nu_2}R_{\mu_2}{}^{\sigma}{}_{\mu\mu_1} = 0$, which is obtained from (56), and in the second line, we have used (57). From the analyses in the previous sections, we know that there exists the constraint $\phi^{(3)}$ if the expression of $\nabla_{\mu}E^{\mu\nu}$ does not include the derivative of h_{00} with respect to time although this is not the necessary condition that there only exist five degrees of freedom. This condition is satisfied in $\nabla_{\mu}E_K^{\mu\nu}$ and also trivially in the contribution from the mass terms and therefore the model does not include ghost even if we set all parameters 0. On the other hand, we may add extra terms with non-minimal coupling if the terms do not induce ghost. This extra terms can be added if we choose $\beta = -\alpha$,

$$\begin{aligned} S &= \int d^4x\sqrt{-g}\left[\frac{1}{2}g^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}\nabla_{\mu_1}h_{\mu_2\nu_2}\nabla_{\nu_1}h_{\mu_3\nu_3} + \frac{m^2}{2}g^{\mu_1\nu_1\mu_2\nu_2}h_{\mu_1\nu_1}h_{\mu_2\nu_2}\right. \\ &\quad \left.+ \frac{\alpha}{2}Rg^{\mu_1\nu_1\mu_2\nu_2}h_{\mu_1\nu_1}h_{\mu_2\nu_2} + \frac{\gamma}{2}C^{\mu\alpha\nu\beta}h_{\mu\nu}h_{\alpha\beta}\right]. \end{aligned} \quad (59)$$

On the Einstein manifold, because the scalar curvature is constant the terms which are proportional to α can be absorbed into the redefinition of the mass terms, which tells that the ghost is not generated on the Einstein manifold. Furthermore, the term proportional to γ change only the coefficient of the first term of the second line in (58) and therefore this term does not induce the ghost.

We now show that the model in (59) does not surely include the ghost. The equation given by the variation of the action (59) is given by

$$E^{\mu\nu} \equiv -g^{(\mu\nu)\mu_1\nu_1\mu_2\nu_2}\nabla_{\mu_1}\nabla_{\nu_1}h_{\mu_2\nu_2} + (m^2 + \alpha R)g^{\mu\nu\mu_1\nu_1}h_{\mu_1\nu_1} + \gamma C^{\mu\alpha\nu\beta}h_{\alpha\beta} = 0. \quad (60)$$

Then we obtain the primary constraint $E^{0\mu} \equiv \phi^{(1)\mu} \approx 0$. The secondary constraint is given by

$$\nabla_{\mu}E^{\mu\nu} = (\gamma - 1)C^{\mu\alpha\nu\beta}\nabla_{\mu}h_{\alpha\beta} + \left\{m^2 + R\left(\frac{1}{6} + \alpha\right)\right\}g^{\mu\nu\alpha\beta}\nabla_{\mu}h_{\alpha\beta} + \gamma\nabla_{\mu}C^{\mu\alpha\nu\beta} \cdot h_{\alpha\beta} \equiv \phi^{(2)\nu} \approx 0. \quad (61)$$

The last term $\nabla_{\mu}C^{\mu\alpha\nu\beta} \cdot h_{\alpha\beta}$ vanishes on the Einstein manifold by the Bianchi identity. The second order derivative of $\phi^{(2)i}$ with respect to time does not contain the second order derivative of h_{00} with respect to time and the constraint (61) determines the values of $\nabla_0\nabla_0h_{0i}$. We also obtain

$$\nabla_{\mu}\nabla_{\nu}E^{\mu\nu} - (\gamma - 1)C^{0i0j}A^{-1}_{ij}{}^{,kl}E_{kl} + \frac{1}{2}\left\{m^2 + R\left(\frac{1}{6} + \alpha\right)\right\}g^{\alpha\beta}E_{\alpha\beta}$$

$$\begin{aligned}
&= -(\gamma - 1)C^{i(0j)\beta}\nabla_0\nabla_j h_{i\beta} - (\gamma - 1)C^{0i0j}A^{-1}_{ij},{}^{kl}(-2g_{kl}^{(0c)\mu\nu})\nabla_0\nabla_c h_{\mu\nu} \\
&+ (\text{terms which do not include } \partial_0 h) \equiv \phi^{(3)} \approx 0.
\end{aligned} \tag{62}$$

Here A^{-1} is defined by (34). Eq. (62) does not contain the second order derivative of h_{00} and other terms including the second order derivative with respect to time can be eliminated by the equations which we have already obtained. Therefore we obtain one more constraint and the system has five degrees of the freedom and there does not appear a ghost.

Finally we investigate the relation between (59) and the non-minimal coupling in [16], which is given by

$$\frac{\xi}{4}Rh_{\alpha\beta}h^{\alpha\beta} + \frac{1-2\xi}{8}Rh^2 = \frac{R}{4}h_{\alpha\beta}h^{\alpha\beta} + \frac{R}{8}h^2 - \frac{\xi-1}{4}Rg^{\mu_1\nu_1\mu_2\nu_2}h_{\mu_1\nu_1}h_{\mu_2\nu_2}. \tag{63}$$

The first two terms correspond to the shift of the mass term. By comparing (63) with the non-minimal coupling terms (53) and (52), we find that the expression (63) corresponds to the case that $\gamma = 1$ in (52). Then we find that in addition to the minimal coupling in [16], we can add the following non-minimal coupling,

$$\frac{\gamma}{2}C^{\mu\alpha\nu\beta}h_{\mu\nu}h_{\alpha\beta}. \tag{64}$$

This term vanish on the (anti-)de Sitter space-time, which is conformally flat, but this term gives non-trivial contribution on the Schwarzschild (anti-)de Sitter space-time, etc.

VII. DERIVATIVE INTERACTION

In [14], it has been shown that there are other kind of pseudo linear terms including derivative interaction although these terms do not correspond to any term in fully non-linear theory [19],

$$l\eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}\partial_{\mu_1}\partial_{\nu_1}h_{\mu_2\nu_2}\cdot h_{\mu_3\nu_3}h_{\mu_4\nu_4}. \tag{65}$$

In this section, when we consider the coupling with gravity, we show that these terms always generate ghost by investigating if the derivative of h_{00} with respect to time could appear in $\phi^{(2)\nu}$.

If we include the derivative interaction terms, there appear the terms proportional to the curvature in the constraint $\phi^{(2)\nu}$ due to the non-commutability of the covariant derivatives. In these terms, there appear the terms including the derivative of h_{00} with respect to time and we need to cancel the terms by including the terms with non-minimal coupling to the action. Because there are not so many types of the non-minimal couplings, however, it is not trivial if we can cancel the terms including the derivative of h_{00} with respect to time and in fact, we fail to cancel the terms.

The derivative interaction term,

$$lg^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}\nabla_{\mu_1}\nabla_{\nu_1}h_{\mu_2\nu_2}\cdot h_{\mu_3\nu_3}h_{\mu_4\nu_4}, \tag{66}$$

give the following contribution to the equation,

$$E_D^{\mu\nu} \equiv 2lg^{(\mu\nu)\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}\nabla_{\mu_1}\nabla_{\nu_1}h_{\mu_2\nu_2}\cdot h_{\mu_3\nu_3} + lg^{(\mu\nu)\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}\nabla_{\mu_1}h_{\mu_2\nu_2}\cdot \nabla_{\nu_1}h_{\mu_3\nu_3}. \tag{67}$$

In the expression of $\nabla_\nu E_D^{\mu\nu}$, the terms including the derivative of h_{00} with respect to time are given by

$$\nabla_\nu E_D^{\mu\nu} \supset lg^{\mu\nu\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}\left(-\frac{1}{2}\nabla_{\nu_2}h_{\mu_2\sigma} + \nabla_\sigma h_{\mu_2\nu_2} - \nabla_{\mu_2}h_{\sigma\nu_2}\right)R_{\mu_1}{}^\sigma{}_{\nu\nu_1}h_{\mu_3\nu_3}. \tag{68}$$

In the first term in the parentheses () in the r.h.s. of (68), we have used (56). We now have following identities,

$$\begin{aligned}
g^{\mu\nu\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}C_{\mu_1}{}^\sigma{}_{\nu\nu_1} &= -6(C^{\nu_2\sigma(\mu\mu_2}g^{\mu_3)\nu_3} + C^{\nu_3\sigma(\mu\mu_3}g^{\mu_2)\nu_2}), \\
g^{\mu\nu\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}g_{\mu_1\nu}{}^\sigma{}_{\nu_1} &= -2g^{\mu\sigma\mu_2\nu_2\mu_3\nu_3},
\end{aligned} \tag{69}$$

In the first equation of (69), the parentheses () does not means the symmetrization but summing up by changing the indices in cyclic way, for example,

$$T_{(\alpha\beta\gamma)} \equiv \frac{1}{3}(T_{\alpha\beta\gamma} + T_{\beta\gamma\alpha} + T_{\gamma\alpha\beta}). \tag{70}$$

By substituting (69) into (68) and using (54), we find that the terms proportional to $g^{\mu\sigma\mu_2\nu_2\mu_3\nu_3}$ are pseudo-linear and therefore do not include the derivative of h_{00} with respect to time. On the other hand, the terms including the Weyl tensor include the derivative of h_{00} with respect to time and have the following forms,

$$\nabla_\nu E_D^{\mu\nu} \supset l \left\{ -C^{\mu\alpha 0\beta} g^{00} + C^{\alpha 0\beta 0} g^{\mu 0} + C^{\mu 00\alpha} g^{\beta 0} \right\} h_{\alpha\beta} \nabla_0 h_{00}. \quad (71)$$

Therefore in order to eliminate the ghost, we need to cancel the terms by including the terms with the non-minimal couplings but if we assume that the terms including the non-minimal couplings could have the following form,

$$c_1 C^{\mu\alpha\nu\beta} h_{\mu\nu} h_{\alpha\beta} h + c_2 C^{\mu\alpha\nu\beta} h_{\mu\nu} h_\alpha^\lambda h_{\lambda\beta}. \quad (72)$$

Then the contribution to $\nabla_\nu E^{\mu\nu}$ from the term (72) are given by

$$\nabla_\mu E^{\mu\nu} \supset \{ (2c_1 + c_2) C^{\mu\alpha 0\beta} g^{00} + (2c_1 + c_2) C^{0\alpha 0\beta} g^{\mu 0} \} h_{\alpha\beta} \nabla_0 h_{00} + (\text{terms not including } \nabla_0 h_{00}), \quad (73)$$

which tells that there cannot be cancellation. Therefore at least in the present formulation we cannot obtain the theory without ghost on the general Einstein manifold. We should note, however, that on the conformally flat space-time, where $C^{\mu\alpha\nu\beta} = 0$, Eq. (68) has the following form,

$$\nabla_\nu E_D^{\mu\nu} \supset -\frac{R}{12} g^{\mu\nu\mu_2\nu_2\mu_3\nu_3} \nabla_\nu h_{\mu_2\nu_2} h_{\mu_3\nu_3}, \quad (74)$$

and we obtain the ghost-free theory even if we do not include the non-minimal coupling because Eq. (74) does not include $\nabla_0 h_{00}$.

We also have tried to eliminate the ghost by including the contribution from the non-minimal coupling in $\nabla_\mu \nabla_\nu E_D^{\mu\nu}$ but we have not succeeded to construct $\phi^{(3)}$ on the general Einstein manifold.

VIII. VARIOUS NON-MINIMAL COUPLINGS

We find that there are many kinds of minimal couplings which do not change the degrees of freedom and do not generate a ghost. Because the scalar curvature is constant on the Einstein manifold and the terms without derivative do not violate the constraints, rather trivial terms are given by

$$R^m g^{\mu_1\nu_1\cdots\mu_n\nu_n} h_{\mu_1\nu_1} \cdots h_{\mu_n\nu_n}. \quad (75)$$

In the previous sections, we also found the term proportional to the Weyl tensor in (64). By using the Weyl tensor, we can construct various non-minimal couplings which do not generate the ghost. For example, if we add the term in (72) to the action which includes only quadratic terms and also do not include ghost, as clear from (73), if we choose $c_2 = -2c_1$, the non-minimal coupling (72) does not change the degrees of freedom and do not generate the ghost. When we neglect the over all factor, the terms of the non-minimal couplings are given by

$$\begin{aligned} & C^{\mu\alpha\nu\beta} h_{\mu\nu} h_{\alpha\beta} h - 2C^{\mu\alpha\nu\beta} h_{\mu\nu} h_{\alpha\lambda} h_\beta^\lambda \\ &= (C^{\mu_1\mu_2\nu_1\nu_2} g^{\mu_3\nu_3} + C^{\mu_1\mu_2\nu_2\nu_3} g^{\mu_3\nu_1} + C^{\mu_1\mu_2\nu_3\nu_1} g^{\mu_3\nu_2}) h_{\mu_1\nu_1} h_{\mu_2\nu_2} h_{\mu_3\nu_3} \\ &= \frac{1}{2 \cdot 3!} \delta_{\rho_1}^{\mu_1} \delta_{\rho_2}^{\mu_2} \delta_{\rho_3}^{\mu_3} \delta_{\sigma_1}^{\nu_1} \delta_{\sigma_2}^{\nu_2} \delta_{\sigma_3}^{\nu_3} C^{\rho_1\rho_2\sigma_1\sigma_2} g^{\rho_3\sigma_3} h_{\mu_1\nu_1} h_{\mu_2\nu_2} h_{\mu_3\nu_3}. \end{aligned} \quad (76)$$

In (76), under the exchange of the indices, the tensor $\delta_{\rho_1}^{\mu_1} \delta_{\rho_2}^{\mu_2} \delta_{\rho_3}^{\mu_3} \delta_{\sigma_1}^{\nu_1} \delta_{\sigma_2}^{\nu_2} \delta_{\sigma_3}^{\nu_3} C^{\rho_1\rho_2\sigma_1\sigma_2} g^{\rho_3\sigma_3}$ has a structure of symmetry which is similar to that of $g^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}$ in the pseudo linear theory as in (64). On the Einstein manifold, because $R_{\mu\nu}$ is proportional to $g_{\mu\nu}$, the terms including $R_{\mu\nu}$ is proportional to the tensor $g^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}$ but because the Weyl tensor do not proportional to the tensor including $g_{\mu\nu}$, we can make a non-trivial tensor. If we only include the terms proportional the first power of the curvature in the non-minimal couplings, the possible tensor for the coefficients could be given by using one Weyl tensor $C^{\mu\alpha\nu\beta}$ and n $g^{\mu\nu}$ as follows,

$$\begin{aligned} & \delta_{\rho_1}^{\mu_1} \delta_{\rho_2}^{\mu_2} \cdots \delta_{\rho_{n+2}}^{\mu_{n+2}} \delta_{\sigma_1}^{\nu_1} \delta_{\sigma_2}^{\nu_2} \cdots \delta_{\sigma_{n+2}}^{\nu_{n+2}} C^{\rho_1\rho_2\sigma_1\sigma_2} g^{\rho_3\sigma_3} \cdots g^{\rho_{n+2}\sigma_{n+2}} \\ & \sim \delta_{\rho_1}^{\mu_1} \delta_{\rho_2}^{\mu_2} \cdots \delta_{\rho_{n+2}}^{\mu_{n+2}} \delta_{\sigma_1}^{\nu_1} \delta_{\sigma_2}^{\nu_2} \cdots \delta_{\sigma_{n+2}}^{\nu_{n+2}} C^{\rho_1\rho_2\sigma_1\sigma_2} g^{\rho_3\sigma_3} \cdots g^{\rho_{n+2}\sigma_{n+2}}. \end{aligned} \quad (77)$$

If we include the higher power of the curvature tensors, we may obtain more kinds of the tensors. In case of the derivative interaction terms, there could occur the difficulties similar to those in the last section even if we includes

the Weyl tensor and possible terms could be given by contracting the indices by using $h_{\mu\nu}$. For the terms without the derivative interactions, it could be manifest that the terms do not generate a ghost. In four dimensions, the possible non-minimal couplings are given by the following three terms,

$$\begin{aligned}
& C^{\mu_1\mu_2\nu_1\nu_2} h_{\mu_1\nu_1} h_{\mu_2\nu_2}, \\
& \delta^{\mu_1}_{\rho_1} \delta^{\mu_2}_{\rho_2} \delta^{\mu_3}_{\rho_3} \delta^{\nu_1}_{\sigma_1} \delta^{\nu_2}_{\sigma_2} \delta^{\nu_3}_{\sigma_3} C^{\rho_1\rho_2\sigma_1\sigma_2} g^{\rho_3\sigma_3} h_{\mu_1\nu_1} h_{\mu_2\nu_2} h_{\mu_3\nu_3} \\
& \delta^{\mu_1}_{\rho_1} \delta^{\mu_2}_{\rho_2} \delta^{\mu_3}_{\rho_3} \delta^{\mu_4}_{\rho_4} \delta^{\nu_1}_{\sigma_1} \delta^{\nu_2}_{\sigma_2} \delta^{\nu_3}_{\sigma_3} \delta^{\nu_4}_{\sigma_4} C^{\rho_1\rho_2\sigma_1\sigma_2} g^{\rho_3\sigma_3\rho_4\sigma_4} h_{\mu_1\nu_1} h_{\mu_2\nu_2} h_{\mu_3\nu_3} h_{\mu_4\nu_4}.
\end{aligned} \tag{78}$$

IX. SUMMARY

In this paper, we consider the model where a new massive spin two model [15] couples with gravity. Although the model proposed in [15] is ghost-free on the Minkowski space-time, the properties on the curved space-time is not so obvious. In fact, Buchbinder et al. have shown that the Fierz-Pauli theory minimally coupled with gravity is not ghost-free [16] and they have obtained a ghost-free theory in the background of the Einstein manifold by adding two non-minimal coupling terms. We also considered the model proposed in [15] on the curved space-time by adding two non-minimal coupling terms as in the case of the Fierz-Pauli theory. Although the calculations become rather complicated and tedious, we have shown that the obtained model does not include ghost. Furthermore we investigated if the derivative interaction on curved space-time can be consistently formulated. Unfortunately, the derivative term induces a ghost even if we take the non-minimal coupling terms into account. Hence, the method of constructing a new spin two theory with the anti-symmetric tensor on Minkowski space-time cannot be extended to the theory on curved space-time by simply replacing $\eta^{\mu\nu}$ with $g^{\mu\nu}$. On the other hand, this pseudo-linear approach leads to the discovery of new non-minimal coupling terms.

A motivation to consider this model on the curved background is applications to the cosmology and black hole physics. When we consider the cosmology, usually we assume the homogeneity and the isotropy of the spacial part of the universe. Furthermore in order to generate the accelerating expansion of the universe, we often consider the condensation of the field like inflaton. In case of the scalar field model, the condensation of the scalar field does not violate the isotropy although the condensation of the abelian vector field violates the isotropy. In fact, the condensation of the spacial components A_i ($i = 1, 2, 3$) in the vector field makes a special direction in the space. On the other hand, the condensation of the temporal component A_0 often conflicts with the gauge invariance because we can usually choose the gauge condition where the temporal component vanishes if there remains the gauge symmetry. In case of the non-abelian gauge theory, however, the condensation of the vector field does not conflict with the isotropy. Non-abelian gauge symmetry always include $SU(2)$ or $SO(3)$ gauge symmetry as a subgroup. Then we may consider the condensation of the vector field $A_i^a = A\delta_i^a$, where $a = 1, 2, 3$ is the index of $SU(2)$ or $SO(3)$. The condensation breaks both of the rotational symmetry, which is $SO(3)$ symmetry and $SU(2)$ or $SO(3)$ gauge symmetry simultaneously because the condensation is not invariant under the rotation, $A_i^{a'} \equiv O_{Ri}^j A_j^a \neq A_i^a$, nor gauge transformation with a constant parameter, $A_i^{a'} \equiv O_G^a{}_b A_i^b \neq A_i^a$, where O_{Rj}^i and $O_G^a{}_b$ are elements of $SO(3)$, corresponding to the rotation and the gauge transformation with a constant parameter, respectively. We should note that the diagonal symmetry is preserved. In fact, the condensation of the vector field is invariant $A_i^{a'} \equiv O_{Ri}^j O_G^a{}_b A_j^b = A_j^a$ if we choose O_R to be equal to the matrix of O_G . Then we may identify this diagonal symmetry as a new rotational symmetry and the isotropy of the spacial part is preserved. On the other hand, in case of the massive spin two field, which is the rank 2 symmetric tensor, the condensation of the trace part (or (t, t) component, or the trace of the spacial part) does not violate the isotropy. Therefore we can consider easily the condensation of the rank 2 symmetric tensor in order to generate the expansion of the universe.

In case of the massive gravity model, by using the condensation, cosmology has been investigated by considering the decoupling limit in [20] and there have been many works about the cosmology in the massive gravity models [10, 21–23] and in the bigravity models [24–34]. It could be also interesting to investigate the black hole entropy as in [35, 36].

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Appendix A: Properties of $g^{\mu_1\nu_1\cdots\mu_n\nu_n}$

In this appendix, we list the properties of $g^{\mu_1\nu_1\cdots\mu_n\nu_n}$. Note that the properties below are held on arbitrary space-time.

1. Definition

First, we define the tensor $g^{\mu_1\nu_1\cdots\mu_n\nu_n}$ as

$$\begin{aligned} g^{\mu_1\nu_1\cdots\mu_n\nu_n} &\equiv g^{\mu_1\nu_1} g^{\mu_2\nu_2} g^{\mu_3\nu_3} \cdots g^{\mu_n\nu_n} - g^{\mu_1\nu_2} g^{\mu_2\nu_1} g^{\mu_3\nu_3} \cdots g^{\mu_n\nu_n} + \cdots \\ &= \frac{-1}{(D-n)!} E^{\mu_1\mu_2\cdots\mu_n\sigma_{n+1}\cdots\sigma_D} E^{\nu_1\nu_2\cdots\nu_n\sigma_{n+1}\cdots\sigma_D}. \end{aligned} \quad (\text{A1})$$

Here D denotes the dimension of the space-time and the totally anti-symmetric tensor $E^{\mu_1\mu_2\cdots\mu_n}$ is defined as

$$E^{\mu_1\mu_2\cdots\mu_D} \equiv \frac{1}{\sqrt{-g}} \epsilon^{\mu_1\mu_2\cdots\mu_D}. \quad (\text{A2})$$

with the totally anti-symmetric Levi-Civita tensor density

$$\epsilon^{\mu_1\mu_2\cdots\mu_D} = \begin{cases} +1 & \text{if } (\mu_1\mu_2\cdots\mu_D) \text{ is an even permutation of } (0123\cdots) \\ -1 & \text{if } (\mu_1\mu_2\cdots\mu_D) \text{ is an odd permutation of } (0123\cdots) \\ 0 & \text{otherwise} \end{cases}$$

In the following, we call the tensor $g^{\mu_1\nu_1\cdots\mu_n\nu_n}$ the pseudo-linear tensor.

Finally, we summarize the symmetric property of the pseudo-linear tensor.

$$\begin{aligned} \mu_i &\longleftrightarrow \mu_j : \text{anti-symmetric} \\ \nu_i &\longleftrightarrow \nu_j : \text{anti-symmetric} \\ (\mu_i, \nu_i) &\longleftrightarrow (\mu_j, \nu_j) : \text{symmetric} \\ \{\mu_i\} &\longleftrightarrow \{\nu_i\} : \text{symmetric} \end{aligned} \quad (\text{A3})$$

2. Useful relations

The contraction of a pair of indices μ_n and ν_n leads to the following relation:

$$g^{\mu_1\nu_1\cdots\mu_{n-1}\nu_{n-1}\mu_n}_{\mu_n} = (D-n+1)g^{\mu_1\nu_1\cdots\mu_{n-1}\nu_{n-1}}. \quad (\text{A4})$$

The pseudo linear tensor can be expanded in terms of the lower rank tensor:

$$\begin{aligned} g^{\mu_1\nu_1\cdots\mu_n\nu_n} &= \delta^{\nu_1}_{\lambda_1} \delta^{\nu_2}_{\lambda_2} \cdots \delta^{\nu_n}_{\lambda_n} g^{\mu_1\lambda_1} \cdots g^{\mu_n\lambda_n} \\ &= \delta^{\nu_1}_{\lambda_1} \delta^{\nu_2}_{\lambda_2} \cdots \delta^{\nu_n}_{\lambda_n} \frac{1}{m!(n-m)!} g^{\mu_1\lambda_1\cdots\mu_m\lambda_m} g^{\mu_{m+1}\lambda_{m+1}\cdots\mu_n\lambda_n}. \end{aligned} \quad (\text{A5})$$

For example,

$$\begin{aligned} g^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3} &= g^{\mu_1\nu_1} g^{\mu_2\nu_2} g^{\mu_3\nu_3} + g^{\mu_1\nu_2} g^{\mu_2\nu_3} g^{\mu_3\nu_1} + g^{\mu_1\nu_3} g^{\mu_2\nu_1} g^{\mu_3\nu_2}, \\ g^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4} &= g^{\mu_1\nu_1} g^{\mu_2\nu_2} g^{\mu_3\nu_3} g^{\mu_4\nu_4} - g^{\mu_1\nu_2} g^{\mu_2\nu_3} g^{\mu_3\nu_4} g^{\mu_4\nu_1} - g^{\mu_1\nu_3} g^{\mu_2\nu_4} g^{\mu_3\nu_1} g^{\mu_4\nu_2} - g^{\mu_1\nu_4} g^{\mu_2\nu_1} g^{\mu_3\nu_2} g^{\mu_4\nu_3}, \end{aligned} \quad (\text{A6})$$

(A4) and (A5) can be easily proven from (A1).

We obtain other useful relations using the ADM variables e_{ij} , N , and N_i :

$$g^{0j i_1 j_1 i_2 j_2 \cdots i_n j_n} = \frac{1}{n!} \delta^j_{k_1} \delta^{j_1}_{k_2} \cdots \delta^{j_n}_{k_n} \frac{N^k}{N^2} e^{i_1 k_1 i_2 k_2 \cdots i_n k_n}, \quad (\text{A7})$$

$$g^{0j_1 i_1 0 i_2 j_2 i_3 j_3 \cdots i_n j_n} = \frac{1}{N^2} e^{i_1 j_1 i_2 j_2 \cdots i_n j_n}. \quad (\text{A8})$$

Here $e^{i_1 j_1 i_2 j_2 \dots i_n j_n}$ is anti-symmetrization of the product $e^{i_1 j_1} e^{i_2 j_2} \dots e^{i_n j_n}$ with respect to j_i .

$$e^{i_1 j_1 i_2 j_2 \dots i_n j_n} \equiv e^{i_1 j_1} e^{i_2 j_2} \dots e^{i_n j_n} - e^{i_1 j_2} e^{i_2 j_1} \dots e^{i_n j_n} + \dots \quad (\text{A9})$$

Let us prove the identities (A8). Just for convenience, we define the following tensors,

$$\begin{aligned} \tilde{g}^{\mu_1 \nu_1 \dots \mu_n \nu_n} &\equiv \frac{1}{n!} g^{\mu_1 \nu_1 \dots \mu_n \nu_n} = \tilde{\delta}_{\lambda_1 \dots \lambda_n}^{\nu_1 \dots \nu_n} g^{\mu_1 \lambda_1} \dots g^{\mu_n \lambda_n} \\ &= \frac{1}{n!} (g^{\mu_1 \nu_1} g^{\mu_2 \nu_2} g^{\mu_3 \nu_3} \dots g^{\mu_n \nu_n} - g^{\mu_1 \nu_2} g^{\mu_2 \nu_1} g^{\mu_3 \nu_3} \dots g^{\mu_n \nu_n} + \dots), \\ \tilde{e}^{i_1 j_1 \dots i_n j_n} &\equiv \frac{1}{n!} e^{i_1 j_1 \dots i_n j_n} = \tilde{\delta}_{k_1 \dots k_n}^{j_1 \dots j_n} e^{i_1 k_1} \dots e^{i_n k_n}. \end{aligned} \quad (\text{A10})$$

Therefore, we can easily prove (A7) as follows,

$$\begin{aligned} \tilde{g}^{0 j_1 i_1 i_2 j_2 \dots i_n j_n} &= \tilde{\delta}_{\lambda}^j \tilde{\delta}_{\lambda_1 \dots \lambda_n}^{j_1 \dots j_n} g^{0 \lambda} g^{i_1 \lambda_1} \dots g^{i_n \lambda_n} = \tilde{\delta}_k^j \tilde{\delta}_{k_1 \dots k_n}^{j_1 \dots j_n} g^{0 k} g^{i_1 k_1} \dots g^{i_n k_n} \\ &= \tilde{\delta}_k^j \tilde{\delta}_{k_1 \dots k_n}^{j_1 \dots j_n} \frac{N^k}{N^2} \left(e^{i_1 k_1} - \frac{N^{i_1} N^{k_1}}{N^2} \right) \dots \left(e^{i_n k_n} - \frac{N^{i_n} N^{k_n}}{N^2} \right) \\ &= \tilde{\delta}_k^j \tilde{\delta}_{k_1 \dots k_n}^{j_1 \dots j_n} \frac{N^k}{N^2} e^{i_1 k_1} \dots e^{i_n k_n} = \tilde{\delta}_k^j \tilde{\delta}_{k_1 \dots k_n}^{j_1 \dots j_n} \frac{N^k}{N^2} \tilde{e}^{i_1 k_1 \dots i_n k_n}. \end{aligned} \quad (\text{A11})$$

Furthermore, we can prove (A8) by using mathematical induction.

1. $n = 1$ case

$$g^{0 j i 0} = \frac{N^i}{N^2} \frac{N^j}{N^2} - \frac{1}{N^2} \left(e^{ij} - \frac{N^i N^j}{N^2} \right) = \frac{e^{ij}}{N^2}. \quad (\text{A12})$$

2. $n = m$ case

If we assume,

$$g^{0 j_1 i_1 0 i_2 j_2 i_3 j_3 \dots i_{m-1} j_{m-1}} = \frac{1}{N^2} e^{i_1 j_1 i_2 j_2 \dots i_{m-1} j_{m-1}}, \quad (\text{A13})$$

then we find

$$\begin{aligned} \tilde{g}^{0 j_1 i_1 0 i_2 j_2 \dots i_m j_m} &= \tilde{\delta}_{\lambda_1}^{j_1} \tilde{\delta}_{\lambda}^0 \tilde{\delta}_{\lambda_2 \dots \lambda_m}^{j_2 \dots j_m} g^{i_m \lambda_m} \tilde{g}^{0 \lambda_1 i_1 \lambda_2 i_2 \dots i_{m-1} \lambda_{m-1}} \\ &= \frac{1}{m+1} \left[g^{i_m j_m} \tilde{g}^{0 j_1 i_1 0 i_2 j_2 \dots i_{m-1} j_{m-1}} - g^{i_m j_1} \tilde{g}^{0 j_m i_1 0 i_2 j_2 \dots i_{m-1} j_{m-1}} \right. \\ &\quad \left. - g^{i_m 0} \tilde{g}^{0 j_1 i_1 j_m i_2 j_2 \dots i_{m-1} j_{m-1}} - g^{i_m j_2} \tilde{g}^{0 j_1 i_1 0 i_2 j_m \dots i_{m-1} j_{m-1}} \dots \right. \\ &\quad \left. - g^{i_m j_{m-1}} \tilde{g}^{0 j_1 i_1 0 i_2 j_2 \dots i_{m-1} j_m} \right] \\ &= \frac{1}{m+1} \left[m \tilde{\delta}_{k_1}^{j_1} \tilde{\delta}_{k_2 \dots k_m}^{j_2 \dots j_m} g^{i_m k_m} \tilde{g}^{0 k_1 i_1 0 i_2 k_2 \dots i_{m-1} k_{m-1}} - g^{i_m 0} \tilde{g}^{0 j_1 i_1 j_m i_2 j_2 \dots i_{m-1} j_{m-1}} \right] \\ &= \frac{1}{m+1} \left[\tilde{\delta}_{k_1}^{j_1} \tilde{\delta}_{k_2 \dots k_m}^{j_2 \dots j_m} g^{i_m k_m} \frac{1}{N^2} \tilde{e}^{i_1 k_1 i_2 k_2 \dots i_{m-1} k_{m-1}} - g^{i_m 0} \tilde{g}^{0 j_1 i_1 j_m i_2 j_2 \dots i_{m-1} j_{m-1}} \right] \\ &= \frac{1}{m+1} \left[\tilde{\delta}_{k_1}^{j_1} \tilde{\delta}_{k_2 \dots k_m}^{j_2 \dots j_m} e^{i_m k_m} \frac{1}{N^2} \tilde{e}^{i_1 k_1 i_2 k_2 \dots i_{m-1} k_{m-1}} \right] \\ &= \frac{1}{m+1} \frac{1}{N^2} \tilde{e}^{i_1 j_1 i_2 j_2 \dots i_m j_m}. \end{aligned} \quad (\text{A14})$$

We used the assumption (A13) in the fourth line and also used equation (A7) in the fifth line.

So we have proved equations (A7) and (A8).

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